Content

Tutorial 11
---Chan Ki Fung

BACK

Questions of today

1. a. Let γ be a positively oriented Jordan curve (simpe closed curve), and Ω be the region enclosed by γ, show that

$$\text{Area}(\Omega) = \frac{1}{2i}\int_\gamma \bar{z}dz$$

   b. (Area theorem) Suppose f is holomorphic and injective on $\mathbb{D}\setminus\{0\}$ and has the power series representation

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

   then show that

$$\sum_{n=0}^{\infty} n|a_n|^2 \le 1.$$

   In particular, we must have $|a_1| \le 1$.

2. a. Suppose f is holomorphic and injective on $\mathbb{D}$ with

$$f(0) = 0, f'(0) = 1.$$

   Show that there exists a function g which holomorphic and injective on $\mathbb{D}$ with

$$g(0) = 0, g'(0) = 1.$$

   and such that $g^2(z) = f(z^2)$.

   b. Suppose f is holomorphic and injective on $\mathbb{D}$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

   show that $|a_2| \le 2$ and $f(\mathbb{D}) \supset D(0, \frac{1}{4})$.

   c. Suppose F is holomorphic and injective on $\mathbb{D}\setminus\{0\}$, and F has a pole of order 1 at $z = 0$, with residue 1. Show that if $w_1, w_2 \notin F(\mathbb{D})$, then $|w_1 - w_2| \le 4$.

3. For $0 \le r < R \le \infty$, let $A(r, R)$ be the annulus $\{z \in \mathbb{C} : r < |z| < R\}$. Show that $A(r_1, R_1)$ and $A(r_2, R_2)$ are conformally equivalent if and only if $R_2/R_1 = r_2/r_1$.

Hints & solutions of today

1. a. Apply the Green' theorem (writing $\bar{z} = x - iy, dz = da + idy$).

   b. Show that

$$\frac{1}{r^2} \le \sum_{n=0}^{\infty} n|a_n|r^{2n}$$

   by applying (a) to the curve $f(C_r)$, where C_r is a circle centered at the origin of radius r. Then take $r \to 1$.

2. a. Show that $f(z) = z\phi(z)$ with ϕ nowhere vanishing, choose holomorphic h such that $h^2(z) = \phi(z)$. Then take $g(z) = zh(z^2)$.
   To show $g'(0) = 1$, find the first few terms of its power series expansion. To argue g is inective, suppose $g(z) = g(w)$, use the injectivity of f to show that $z^2 = w^2$. If $z = -w$, use $g(z) = zh(z^2)$ to get $z = 0$.

   b. To show $|a_2| \le 2$, we use part (a) to find g with $g^(z) = f(z^2)$. Show that we have Laurent series expansion:

$$\frac{1}{g(z)} = \frac{1}{z} - \frac{a_2}{2}z + \cdots,$$

   then apply problem 1(b).
   To show the second part, suppose w is not in the image, and put

$$h(z) = \frac{wf(z)}{w - f(z)}.$$

   Show that h is holomorphic and injective and so that

$$h(z) = z + (a_2 + \frac{1}{w})z^2 + \cdots.$$

   Hence $|a_2 + \frac{1}{w}| \le 2$ if we applied what we have proved for f to h. Hence $\frac{1}{|w|} \le 4$.

   c. Consider

$$f(z) = \frac{1}{F(z) - w_1},$$

   Show that f satisfies the assumption of part (b). Thus we must have

$$\frac{1}{|w_2 - w_1|} \ge \frac{1}{4}.$$

3. The "if" direction is easy. For the only if direction, we can consider the case $r_1 = r_2 = 1$, so we need to show that if $A(1, R_1)$ and $A(1, R_2)$ are conformally equivalent, then $R_1 = R_2$. We divide the hints into several steps.
   Step 1: Suppose $f : A(1, R_1) \to A(1, R_2)$ is a conformal equivalence. For $1 < r < R_1$, let C_r be the circle of radius r centered at the origin. Show that there exists some small positive ϵ such that $f(A(1, 1+\epsilon)) \cap f(C_r) = \emptyset$. Replace f with R_2/f if necessary, we may assume $f(A(1, 1+\epsilon)) \subset A(1, r)$.
   Step 2: Taking $r \to 1$, we see that $|f(z)| \to 1$ as $|z| \to 1$. In the same manner, show that $|f(z)| \to R_2$ as $|z| \to R_1$.
   Step 3: Consider the function

$$u(z) = \log|f(z)| - t\log|z|,$$

   where t is a real number. Note that u is harmonic, show that for some suitable t, u becomes 0 on the boundary of $A(1, R_1)$, and thus u is identically zero by the harmonicity.
   Step 4: From step 3, we see that $f/f' = t/z$, show that t is an integer using argument principle, and show that it is positive.
   Step 5: From step 4, we have $f = cz^t$. Show that $|c| = t = 1$.
